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The Representation of Seismic Waves in Frequency-Wave Number Space

E. J. Kelly, Jr.

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THE REPRESENTATION OF SEISMIC WAVES IN FREQUENCY-WAVE NUMBER SPACE

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ABSTRACT

In seismic discrimination problems one is interested in filtering out information carried in certain waves from noise carried in other waves. With the use of arrays of sensors, this filtering can be performed in space and time. It is quite useful to visualize the filtering in the Fourier-conjugate space of wave number and frequency, and this point of view is developed here. Most of the report is devoted to an expression of the pertinent facts about the propagation of elastic waves in the earth in terms of the frequency-wave number diagram. The remainder concerns signal and noise models and the interpretation of filtering techniques in frequency-wave number space.

This technical documentary report is approved for distribution.

Franklin C. Hudson

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I. INTRODUCTION

The successful development of statistical decision procedures for seismic data processing depends directly on the accuracy with which the signal and noise models employed reflect the properties of the actual seismic waves. In detail these models must be empirically determined, but there are certain general principles to which they must adhere, and a study of these principles themselves provides valuable insight into the problem of model construction. The general principles follow in turn from the fact that seismic signals and noise are elastic waves propagating in a nearly spherically symmetric, radially stratified medium, and that they are produced and observed at (or very near) the surface.

The salient features of the elastic properties of the earth are its high degree of spherical symmetry and the fact that the continuous variation of propagation speeds with radial distance is interrupted by a number of discontinuities, giving the medium a spherically layered structure. The discontinuity which separates the mantle from the core (at a radius of 0.548R, where R is the mean earth radius) strongly affects the propagation of body waves, and must, of course, be analyzed with due regard for the spherical geometry involved. However the Mohorovičić discontinuity at the base of the crust and various discontinuities within the crust lie very near the surface, and they give rise to waves confined to the immediate vicinity of the surface. As a result, the propagation features associated with these layers are well described by an equivalent plane-layered model earth, consisting of one or more layers overlying an infinite half-space.

A source in the layered region will send energy along the surface in various wave-guide modes, some of which (trapped modes such as Rayleigh and Love waves) confine the energy flux rigorously to the surface, while others (leaking modes such as PL) leak energy slowly down into the half-space. In addition to these waves, the source will send ordinary body waves down into the half-space. We then interpret the half-space as the interior of the earth, and superpose on the plane-layered model the refractive and other effects that will allow energy to re-emerge at the surface.

The theoretical determination of the elastic field produced by a given source distribution constitutes, in general, a very difficult wave-propagation, or diffraction problem.

One must solve the elastic wave equation for prescribed sources and given boundary conditions (at the earth's surface and at the layer interfaces). However, by means of a suitable Green's function, this solution for given sources can be expressed as a superposition of simpler solutions, the eigenfunctions of the problem. These eigenfunctions are generalized plane waves (and they reduce to simple plane waves for an infinite homogeneous medium), with harmonic time-dependence, and they satisfy the boundary conditions as well as the source-free elastic wave equation. The eigenfunctions constitute a three-parameter family of functions. One parameter is frequency (which governs the time-dependence) and the other two are often taken to be the plane-wave direction angles. However, in the seismic problem with its horizontal plane bounding surface, it is more convenient to use azimuth and horizontal wave number (defined below) for parametrization. The representation of an arbitrary elastic field as a superposition of eigenfunctions is analogous to the use of the Fourier integral for functions of time. However, here the analog of the transform function of frequency is a function of the three parameters: frequency, horizontal wave number and azimuth. For remote sources in a plane-layered medium, the role of the azimuth parameter is trivial, and it relates only to the source distribution. However the frequency, ω, and horizontal wave number, κ, are intimately related to the properties of the medium as well as the source, hence the representation of elastic waves in terms of their density in the $\omega - \kappa$ plane provides a convenient focal point for the discussion of the general principles mentioned above. In the case of seismic noise, the appropriate description is in terms of a spectral density in the $\omega - \kappa$ plane.

In the following sections we discuss the form of the basic eigenfunctions and the significance of regions of the $\omega - \kappa$ plane beginning with a simple example and ending with a qualitative discussion of the actual earth. A final section then deals with models of seismic signals and noise, emphasizing the fundamental restrictions imposed by the physical nature of the medium. The discussion is expository throughout, and contains no new results. Sections II, III and IV are intended as aids in the understanding of some of the basic facts of theoretical seismology.* Section V provides the framework within which models can be made and processing techniques can be evolved and analyzed.

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* See References 1 and 2 for further details.

II. ACOUSTIC WAVES IN AN INFINITE HALF-SPACE

Let the scalar quantity, $\varphi(\vec{r},t)$, be the velocity potential of an acoustic field, satisfying the wave equation

$$\nabla^2 \varphi(\vec{r},t) - \frac{1}{c^2} \frac{\partial^2 \varphi(\vec{r},t)}{\partial t^2} = 0 \tag{2.1}$$

The velocity field itself is $\vec{v}(\vec{r},t) = \nabla \varphi(\vec{r},t)$ and the pressure is $p(r,t) = -\rho \frac{\partial \varphi(\vec{r},t)}{\partial t}$, where ρ is the density of the medium.

For an infinite homogeneous space, the basic eigenfunctions are simple plane waves:

$$\varphi(\vec{r},t) = A\, e^{i(\vec{k}\cdot\vec{r}-\omega t)} \tag{2.2}$$

where A is a complex constant and the real and imaginary parts of (2.2) are each solutions of (2.1) provided

$$|\vec{k}|^2 - (\omega/c)^2 = 0 \tag{2.3}$$

According to (2.3), the eigenfunctions are a three-parameter family, which may be indexed by the three components of $\vec{k}$ (in which case $\omega \equiv |\vec{k}|c$), or by ω and the two direction cosines of $\vec{k}$ (in which case $|\vec{k}| \equiv \omega/c$).

Although we always keep the frequency real, Eq. (2.3) does not force the components of $\vec{k}$ to be real. For example, k_3 (the z-component of $\vec{k}$) might be pure imaginary, so that k_3^2 is negative, without violating (2.3). The solution (2.2) would then have the form

$$\varphi(\vec{r},t) = A \exp(\pm|k_3|z) \exp[i(k_1x + k_2y - \omega t)],$$

and would increase exponentially in either the positive or negative z-direction. Since this cannot be permitted in an infinite medium, it can be seen that only $\vec{k}$-vectors with real components are possible solutions of (2.3). Such solutions are called real plane waves.

In the problem of the half-space, we suppose that the boundary is the plane, $z = 0$, and that the medium lies in the upper space, $z \geq 0$. We now form the eigenfunctions of this problem out of the plane waves (2.2). First we solve Eq. (2.3) for k_3:

$$k_3 = \pm \sqrt{\left(\frac{\omega}{c}\right)^2 - \kappa^2} \tag{2.4}$$

where κ, defined by

$$\kappa \equiv \sqrt{k_1^2 + k_2^2} \quad , \tag{2.5}$$

is the "horizontal wave-number." We let $\vec{\kappa}$ be the vector, in the x-y plane, whose components are k_1 and k_2, and whose magnitude is κ. We also let $\vec{\rho}$ be the projection of the position vector, $\vec{r}$, on the x-y plane, so that

$$k_1 x + k_2 y = \vec{\kappa} \cdot \vec{\rho} \tag{2.6}$$

Then the plane wave (2.2) can be written in the form

$$\varphi(\vec{r}, t) = A\, e^{ik_3 z}\, e^{i(\vec{\kappa} \cdot \vec{\rho} - \omega t)} \tag{2.7}$$

which is especially suited to a treatment which emphasizes the values of the field on the boundary plane itself. In fact, we now choose to index the eigenfunctions by the parameters frequency (ω), horizontal wave number (κ) and azimuth (direction of $\vec{\kappa}$).

According to (2.4) there are two plane waves for each set of parameter values. However, only a single combination of these two will fit the boundary condition on the plane $z \equiv 0$. It can be shown that the field is uniquely determined by either of two conditions at the boundary:

a) $\varphi(\vec{r}, t) = 0$ at $z = 0$ (2.8a)

or

b) $\dfrac{\partial \varphi(\vec{r}, t)}{\partial z} = 0$ at $z = 0$ (2.8b)

It follows from condition a) that $\frac{\partial\varphi}{\partial t}$ vanishes at z = 0, hence the pressure is zero on the boundary. This is called a "soft boundary" condition and corresponds to the presence of a free surface. The other condition, (2.8b), says that v_3, the z-component of velocity, vanishes on the boundary, and this is called a "hard boundary" condition, since the boundary surface is not allowed to move. It is also possible to determine a unique solution by means of a mixed boundary condition

$$a\varphi(\vec{r},t) + b\,\frac{\partial\varphi(\vec{r},t)}{\partial z} = 0 \qquad \text{on } z = 0,$$

or equivalently, in terms of pressure and velocity,

$$\frac{p(x,y,0,t)}{v_3(x,y,0,t)} = Z$$

This is called an "impedance boundary condition," and Z is the mechanical independence (per unit area) of the surface. It is useful, for example, when the bounding surface is a piston of finite mass and compliance.

To satisfy any of these boundary conditions for a fixed parameter set $(\omega,\vec{\varkappa})$, a combination of the two real plane waves (2.7) must be used. If we take, for k_3, the positive root

$$k_3 = \sqrt{(\omega/c)^2 - \varkappa^2}\ ,$$

then we obtain the basic eigenfunctions:

a) Soft boundary

$$\varphi(\vec{r},t) = A\,e^{ik_3z}\,e^{i(\vec{\varkappa}\cdot\vec{\rho}-\omega t)} - A\,e^{-ik_3z}\,e^{i(\vec{\varkappa}\cdot\vec{\rho}-\omega t)}$$

$$= (2iA)\sin(k_3z)\,e^{i(\vec{\varkappa}\cdot\vec{\rho}-\omega t)}$$

b) Hard boundary

$$\varphi(\vec{r},t) = A\, e^{ik_3 z}\, e^{i(\vec{\kappa}\cdot\vec{\rho}-\omega t)} + A\, e^{-ik_3 z}\, e^{i(\vec{\kappa}\cdot\vec{\rho}-\omega t)}$$

$$= 2A \cos (k_3 z)\, e^{i(\vec{\kappa}\cdot\vec{\rho}-\omega t)}$$

In these equations, A is complex, and the real and imaginary parts of $\varphi(\vec{r},t)$ provide independent eigenfunctions which differ only in the time origin. For example, in the soft boundary case (which corresponds to the seismic problem) the two real solutions are

$$\sin (k_3 z) \cos (\vec{\kappa}\cdot\vec{\rho}-\omega t) \tag{2.9}$$

and

$$\sin (k_3 z) \sin (\vec{\kappa}\cdot\vec{\rho}-\omega t) \tag{2.9'}$$

The representation in terms of the difference of plane waves shows that each eigenfunction may be thought of as an incident and reflected wave, with equal angles of incidence and reflection, and a reflection coefficient (for velocity) of (− 1). For economy of writing, we use only the soft boundary condition from now on, although perfectly analogous results hold for the hard boundary or (more generally) the impedance boundary condition problems.

It is interesting to derive the particle motion that obtains for one of our eigenfunctions, say (2.9). The component of velocity in the x-y plane is given by

$$-\vec{\kappa} \sin (k_3 z) \sin (\vec{\kappa}\cdot\vec{\rho}-\omega t),$$

which vanishes on the boundary. The "vertical" component, however, is

$$v_3 = \frac{\partial\varphi(\vec{r},t)}{\partial z} = k_3 \cos (k_3 z) \cos (\vec{\kappa}\cdot\vec{\rho}-\omega t)$$

which is a maximum at the boundary. Thus the particle motion at the boundary is linearly polarized, perpendicular to the boundary, and has the form of a wave traveling over the surface with frequency ω, apparent wave number $\varkappa$ (wave length $2\pi/\varkappa$), direction $\vec{\varkappa}$, and horizontal phase velocity $\omega/\varkappa$. Since we always have

$$\varkappa \leq \omega/c \tag{2.10}$$

for real plane waves (real $\vec{k}$-vector), the horizontal phase velocity is never less than the speed of sound.

For each point in the ω-$\varkappa$ space satisfying (2.10), we have an eigenfunction formed from real plane waves. This region clearly forms a cone about the ω-axis, and in the ω-$\varkappa$ plane has the form of the shaded region in Fig. 1. In this figure, the slope of the boundary line is just c, the sound speed in the medium. The shaded region, which contains all eigenfunctions, is called the continuum.

The forbidden region in Fig. 1 is defined by the condition $\varkappa \geq \omega/c$. By Eq. (2.4), this corresponds to plane waves with pure imaginary values of k_3. We have seen that these so-called complex plane waves were excluded in infinite space because they become infinitely large in the $\pm$ z-direction. However, for the half-space, we always have $z \geq 0$, so that the case $k_3 = ip$ where

$$p \equiv +\sqrt{\varkappa^2 - (\omega/c)^2} \tag{2.11}$$

is not excluded. For this case,

$$\varphi(\vec{r},t) = e^{-pz} \cos (\vec{\varkappa}\cdot\vec{\rho}-\omega t) \tag{2.12}$$

and it appears that we have another solution. However, without the other complex plane wave (which has z-dependence e^{+pz}), we cannot fit any of the boundary conditions discussed above. Nevertheless, because of the importance of such waves in the elastic case, it is interesting to derive the associated particle motion. We take the x-axis along $\vec{\varkappa}$, so that the disturbance propagates along the x-axis with phase velocity $(\omega/\varkappa) < c$. Then

$$v_x = \frac{\partial\varphi(\vec{r},t)}{\partial x} = -\varkappa\, e^{-pz} \sin(\varkappa x-\omega t)$$

and

$$v_z = \frac{\partial\varphi(\vec{r},t)}{\partial z} = -p\, e^{-pz} \cos(\varkappa x-\omega t)$$

These components of velocity are out of phase, so that the resultant, although a pure sound wave, is not longitudinal in the ordinary sense. The fact that the z-component of velocity is out of phase with the pressure,

$$p(r,t) = -\rho\,\frac{\partial\varphi(\vec{r},t)}{\partial t} = \rho\omega e^{-pz} \sin(\varkappa x-\omega t)\ ,$$

implies no flux of energy in the z-direction. At a given point on the boundary, the particle trajectory is given by the parametric equations

$$\begin{aligned} x(t) &= -\varkappa \cos \omega(t-t_o) \\ z(t) &= -\,p \sin \omega(t-t_o) \end{aligned} \tag{2.13}$$

The particle motion is elliptical (prograde), with major and minor ellipse axes along the x- and z-axes.

III. ELASTIC WAVES IN AN INFINITE HALF-SPACE

Elastic waves are described by a vector field, $\vec{u}(\vec{r},t)$, which represents the displacement from equilibrium due to the elastic disturbance. Instead of discussing the rather complicated equation which $\vec{u}$ obeys, we make use of that fact that any elastic field may be expressed as the sum of two fields:

$$\vec{u}(\vec{r},t) = \vec{u}_p(\vec{r},t) + \vec{u}_s(\vec{r},t) \tag{3.1}$$

each of which satisfies a simple wave equation:

$$\nabla^2 \vec{u}_p(\vec{r},t) - \frac{1}{\alpha^2} \frac{\partial^2 \vec{u}_p(\vec{r},t)}{\partial t^2} = 0 \tag{3.2a}$$

$$\nabla^2 \vec{u}_s(\vec{r},t) - \frac{1}{\beta^2} \frac{\partial^2 \vec{u}_s(\vec{r},t)}{\partial t^2} = 0 \tag{3.2b}$$

and an auxiliary condition:

$$\nabla \times \vec{u}_p(\vec{r},t) = 0 \tag{3.3}$$

$$\nabla \cdot \vec{u}_s(\vec{r},t) = 0 \tag{3.4}$$

The partial field $\vec{u}_p(\vec{r},t)$ is irrotational, describes dilatational disturbances, and propagates with phase velocity α. Plane waves of this type are called P-waves. The partial field $\vec{u}_s(\vec{r},t)$ is solenoidal, describes equi-voluminal disturbances (pure shear), and propagates with phase velocity $\beta < \alpha$. Plane waves of this type are called S-waves.

It follows from (3.3) that $\vec{u}_p(\vec{r},t)$ can be derived from a scalar potential function, $\psi(\vec{r},t)$:

$$\vec{u}_p(\vec{r},t) = \nabla \psi(\vec{r},t) ,$$

which satisfies the wave equation with velocity α. The time-derivative, $\frac{\partial \psi(\vec{r},t)}{\partial t}$, is entirely equivalent to the velocity-potential, $\varphi(\vec{r},t)$, of Section II. The $\vec{u}_p$ field is therefore identical in properties with the acoustic velocity field discussed above. In infinite space, the basic u_p-waves are real plane waves:

$$\vec{u}_p(\vec{r},t) = \vec{U}_p e^{i(\vec{k}\cdot\vec{r}-\omega t)} , \tag{3.5}$$

where

$$|\vec{k}|^2 - (\omega/\alpha)^2 = 0 \tag{3.6}$$

and the components of $\vec{k}$ are real. For given values of ω and $\varkappa$, there are two waves, $(\vec{u}_p)^{\pm}$, one corresponding to each sign of k_3. According to condition (3.3), which must hold everywhere and for all time, the polarization vector, $\vec{U}_p$, satisfies

$$\vec{k} \times \vec{U}_p = 0 \tag{3.7}$$

Although $\vec{U}_p$ may be complex, it follows that the real and imaginary parts are each parallel to $\vec{k}$, so that $\vec{U}_p$ itself has the form

$$\vec{U}_p = A\,\vec{k} , \tag{3.8}$$

where A is complex (by redefining the time origin, we can make A real). Equation (3.8) expresses the fact that P-waves are longitudinal.

In the half-space in addition to real plane waves, with $\varkappa < \omega/\alpha$, we again consider the complex plane wave, $(\vec{u}_p)^o$, which is obtained if $\varkappa > \omega/\alpha$. We can still write this wave in the form (3.5) if we interpret $\vec{k}$ as a complex vector:

$$\vec{k} = \vec{\varkappa} + ip\,\vec{e}_z \tag{3.9}$$

where $\vec{e}_z$ is a unit vector in the positive z-direction, $\vec{\varkappa}$ is the horizontal component of $\vec{k}$, as before, and

$$p \equiv +\sqrt{\varkappa^2 - (\omega/\alpha)^2} \tag{3.10}$$

The z-dependence of this wave is given by the factor e^{-pz}. Now condition (3.3) is still equivalent to (3.7) (although $\vec{k}$ and $\vec{U}_p$ are both complex), and it may be shown that (3.7) still implies (3.8). In the algebraic sense, the wave is still longitudinal, although the particle motion is quite different from that of a real P-wave. Substituting, we find that

$$(\vec{u}_p)^o = A\vec{k}\, e^{i(\vec{k}\cdot\vec{r}-\omega t)} = A(\vec{\varkappa} + ip\,\vec{e}_z)\; e^{-pt}\; e^{i(\vec{\varkappa}\cdot\vec{\rho}-\omega t)} \quad . \tag{3.11}$$

We choose the time origin to make A real, set it equal to unity, and take the real part of $\vec{u}$. The result is the wave

$$(\vec{u}_p)^o = \vec{\varkappa}\, e^{-pz} \cos(\vec{\varkappa}\cdot\vec{\rho} - \omega t) - p\,\vec{e}_z\, e^{-pz} \sin(\vec{\varkappa}\cdot\vec{\rho} - \omega t) \tag{3.11'}$$

which describes prograde elliptical particle motion. In fact, except for a constant factor, the particle motion described by (3.11') is identical with that obtained in Section II for complex acoustic waves.

In infinite space, the basic u_s-waves are real plane waves:

$$\vec{u}_s = \vec{U}_s\, e^{i(\vec{k}\cdot\vec{r}-\omega t)} \tag{3.12}$$

where

$$|\vec{k}|^2 - (\omega/\beta)^2 = 0 \tag{3.13}$$

and the components of $\vec{k}$ are real. There are two waves, $(\vec{u}_s)^{\pm}$, for each set $(\omega,\vec{\varkappa})$ and for each possible polarization. Condition (3.4) now reduces to

$$\vec{k}\cdot\vec{U}_s = 0 \tag{3.14}$$

which means that $\vec{U}_s$ is an arbitrary complex combination of two independent real vectors, each perpendicular to $\vec{k}$. Thus S-waves have transverse particle motion. In the half-space, it is natural to choose these two directions so that one is perpendicular to both $\vec{k}$ and $\vec{e}_z$, while the other is in the $(k - e_z)$ plane. For simplicity, we take the x-axis along $\vec{\varkappa}$, and make use of the unit vectors $\vec{e}_x$ and $\vec{e}_y$, along with x and y axes, respectively.

Then the first type of S-wave is called an SH-wave, is polarized in the horizontal plane, and has the form

$$\vec{u}_{sh} = A\,\vec{e}_y\, e^{i(\vec{k}\cdot\vec{r}-\omega t)} , \tag{3.15}$$

for either sign of k_3. The other wave is called an SV-wave, is polarized in the vertical plane containing $\vec{k}$, and can be written in the form

$$\vec{u}_{sv} = A(\vec{k} \times \vec{e}_y)\, e^{i(\vec{k}\cdot\vec{r}-\omega t)} \tag{3.16}$$

again for either sign of k_3.

Now in the half-space, in addition to linearly polarized real plane waves, we have the new possibility of complex S-waves, $(\vec{u}_{sh})^o$ and $(\vec{u}_{sv})^o$. We must have $\varkappa > \omega/\beta$ (which exceeds ω/α), and we still use (3.12) with the complex $\vec{k}$-vector:

$$\vec{k} = \vec{\varkappa} + iq\,\vec{e}_z = \varkappa\,\vec{e}_x + iq\,\vec{e}_z , \tag{3.17}$$

where

$$q \equiv +\sqrt{\varkappa^2 - (\omega/\beta)^2} . \tag{3.18}$$

It is not difficult to show that condition (3.4) is still equivalent to (3.14) which is still satisfied for two types of wave, again of the form (3.15) and (3.16) (although there is only one complex SH-wave and one complex SV-wave, since the sign of q is fixed). The complex SH-wave is still truly transverse, although the complex SV-wave is only formally transverse, in the sense that (3.14) remains true. The complex SH-wave has the specific real form

$$(\vec{u}_{sh})^o = \vec{e}_y\, e^{-qz} \cos(\vec{\varkappa}\cdot\vec{\rho} - \omega t) \tag{3.19}$$

with simple harmonic linearly polarized particle motion along the y-axis. The corresponding form for the complex SV-wave is obtained by substituting (3.17) into (3.16) and taking the real part (A may be taken real and ignored):

$$(\vec{u}_{sv})^o = \varkappa\, \vec{e}_z\, e^{-qz} \cos(\varkappa x - \omega t) + q\, \vec{e}_x\, e^{-qz} \sin(\varkappa x - \omega t) \tag{3.20}$$

It may be seen that (3.20), like (3.11'), describes prograde elliptical particle motion (had we taken A to be pure imaginary--a shift of time origin -- Eq. (30) would have come out resembling (3.11') even more closely.) Thus complex P-waves and complex SV-waves are very similar and can be combined without destroying the character of the particle motion.

In infinite space, only the real plane waves can exist (as we have seen, there are three pairs of waves). In the half-space problem, we can expect the boundary condition at $z = 0$ to determine the waves, or combination of waves, which can exist. The elastic soft-boundary condition is that the force exerted by the medium on the boundary surface (the "surface traction") is zero. This force is a vector which, for our plane boundary at $z = 0$, is given in terms of the displacement field by the relation

$$\vec{f}(\vec{\rho}, t) = \lambda\, \vec{e}_z\, (\nabla \cdot \vec{u}) + \mu \frac{\partial \vec{u}}{\partial z} + \mu\, \nabla u_z \tag{3.21}$$

The derivatives of $\vec{u}$ are evaluated at $z = 0$, u_z is the z-component of $\vec{u}$, and the constants λ and μ are the Lamé coefficients which characterize the stress-strain relations of the medium.*

For SH-waves, whether real or complex, $\nabla \cdot u = 0$ and $u_z = 0$, so that $\vec{f}$ is given by $\mu\, \partial\vec{u}/\partial z$ and has only a y-component. Thus the boundary condition, $\vec{f} = 0$, reduces to a single equation in this case (P- and SV-waves with the same value of $\vec{\varkappa}$ do not give rise to surface forces with y-components). If $\varkappa < \omega/\beta$, we have two real SV-waves: $(\vec{u}_{sh})^{\pm}$ and the combination

$$\vec{u} = (\vec{u}_{sh})^{+} + (\vec{u}_{sh})^{-}$$

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* In terms of λ, μ and the density, ρ, the propagation speeds are given by $\alpha^2 = (\lambda + 2\mu)/\rho$, $\beta^2 = \mu/\rho$.

results in $\vec{f} = 0$. Thus a single real SH-wave can exist in the half-space, consisting of an incident wave, $(\vec{u}_{sh})^+$, and a reflected wave, $(\vec{u}_{sh})^-$, with reflection coefficient equal to (+1). If $\varkappa > \omega/\beta$, there is only one complex SH-wave, for which $\vec{f} \neq 0$, hence this wave cannot exist. Thus the ω-$\varkappa$ diagram for SH-waves looks just like that for acoustic waves (Fig. 1) except that the dividing line has a slope equal to the shear velocity, β.

We now consider P- and SV-waves together, since it turns out that the boundary conditions cannot be satisfied with either type alone. Since we are dealing with two phase velocities, α and β, there are three (ω-$\varkappa$) regions to be considered, defined by the relations

$$\varkappa < \omega/\alpha \tag{3.22a}$$

$$\omega/\alpha < \varkappa < \omega/\beta \tag{3.22b}$$

$$\omega/\beta < \varkappa \tag{3.22c}$$

These are shown on Fig. 2. The slope of the line between A and B is α; the other line has slope β. In all cases, the force, f, has two components (in the x and z directions), hence the boundary condition, $\vec{f} = 0$, provides two homogeneous equations to be satisfied by the various wave amplitudes.

In region A, both P- and SV-waves are real, hence there are four waves in all. The general combination

$$\vec{u} = A(\vec{u}_p)^+ + B(\vec{u}_p)^- + C(\vec{u}_{sv})^+ + D(\vec{u}_{sv})^-$$

can be made to satisfy the two boundary equations and at the same time, two of the four coefficients can be chosen arbitrarily. For example, the choice A = 1, B = 0 corresponds to an incident P-wave and reflected P- and SV-waves. The independent combination C = 1, D = 0 corresponds to an incident SV-wave, together with both kinds of reflected waves. The reflection coefficients for the waves of the incident type no longer have magnitude unity, because energy is removed by mode conversion to the other type. However, in each of the two cases given above, the amplitudes of both reflected waves

are real. This results in a linearly polarized particle motion in the vertical plane. Region A corresponds to all possible angles of incidence for P-waves, and the reflected SV-wave travels in a direction closer to the normal to the surface, as in Fig. 3. However, for incident SV-waves, region A corresponds to angles of incidence no greater than θ_c, the critical angle at which the reflected P-wave is parallel to the boundary; see Fig. 4.

In region B of the ω-κ plane, there are two real SV-waves, but only a single, complex, P-wave. The general case is a superposition:

$$\vec{u} = A(\vec{u}_p)^o + B(\vec{u}_{sv})^+ + C(\vec{u}_{sv})^-$$

Again, the boundary equations can be satisfied and at the same time one of the three amplitudes can be chosen arbitrarily. For example, we may take $B = 1$, and we have an incident SV-wave (with angle of incidence greater than θ_c), a real reflected SV-wave (angle of reflection equal to angle of incidence), and a "reflected" complex P-wave. In this case the reflection coefficients, A and C, are complex, although $|C| = 1$. The particle motion is elliptic in the vertical plane.

In region C, we have only two waves: a complex P and a complex SV. The combination

$$u = A(u_p)^o + B(u_{sv})^o$$

describes a complex wave with elliptical particle motion in the vertical plane. The boundary equations are a pair of homogeneous equations in the two amplitudes, A and B. Thus a non-vanishing solutions exists only if the system determinant vanishes, and this amounts to a relation between ω and κ. In fact, this relation simply gives the ratio, $c_R = \omega/\kappa$, as a function of the phase velocities α and β. In other words, surface waves of this type, called Rayleigh waves, can exist only for one value of κ at each frequency. For the homogeneous half-space under study here, this "dispersion relation," giving κ as a function of ω, is linear:

$$\kappa = \omega/c_R,$$

hence, Rayleigh waves have a characteristic velocity and there is actually no dispersion. For all values of α and β, $c_R < \beta$, and for most practical cases, c_R is nearly equal to β.

We can now summarize the state of affairs in terms of two ω-κ diagrams, one for SH-waves and one for both P- and SV-waves. These are shown in Figs. 5 and 6, in which the labels on the lines give their slopes. For all points in the ω-κ plane to the left of the line β, in Fig. 5, we obtain an eigenfunction representing real SH-waves. In the same region of Fig. 6, we get eigenfunctions representing combinations of P- and SV-waves, with at least two waves real. To the right of line β, in Fig. 6, we get a family of eigenfunctions, all of which correspond to points on the line c_R, representing Rayleigh waves, made up of complex P- and SV-waves. All SH-waves are linearly polarized (transversely) in the horizontal plane. Eigenfunctions of the P-SV type are linearly polarized in the vertical plane for points to the left of line α. To the right of line α, the particle motion is elliptic* in the vertical plane. For any eigenfunction in the ω-κ plane, any azimuth is possible. This exhausts the possibilities for the case of a homogeneous elastic half-space.

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* In Rayleigh waves, the P and SV components are separately prograde in partical motion at the surface, although the combination may be retrograde (as in the actual earth).

IV. ELASTIC WAVES IN A LAYERED HALF-SPACE

As mentioned in the Introduction, most of the observed features of seismic wave propagation can be accounted for by a hybrid treatment which consists of ray methods for body-wave phases traveling through the interior of the earth, and the rigorous diffraction theory of plane-layered media for surface waves. The methods of the previous section apply equally well to the case of n plane layers (of arbitrary type and thickness) overlying a half-space, although the analysis is tedious and the results are complicated in detail. Many of the observed features of surface waves can be explained by assuming only one, or at most two, layers. With infinite parallel plane layers, moreover, SH-waves never mix with P—SV-waves; hence the two cases can be handled separately, and two ω-κ diagrams still suffice to describe the possible eigenfunctions. Since the surface-wave or body-wave character of an eigenfunction depends entirely on its behavior in the half-space, one need only plot the line $\omega = \beta\kappa$ on the SH diagram, and the two lines $\omega = \alpha\kappa$ and $\omega = \beta\kappa$ on the P-SV diagram, to separate these two classes of waves. Here, α and β are the P-wave and S-wave velocities, respectively, in the underlying infinite half-space.

First, we discuss SH-waves. If $\kappa < \omega/\beta$, we are in the continuum, and real plane SH-waves can exist in the underlying half-space. Moreover, there are two of these waves, one traveling toward and one away from the surface. In each layer (of finite thickness) two waves can exist as well. Depending on the values of ω and κ (and the properties of the layer), the waves in a given layer will either be real or complex. Since all SH displacements and forces are in a single direction (for fixed ω and $\vec{\kappa}$), the boundary conditions at an interface (continuity of displacement and surface force) amount to just two equations. Again, at the free surface, $z = 0$, the boundary condition yields one equation. Thus, for the case of n layers over a half-space, there are $2n + 2$ wave amplitudes and a total of $2n + 1$ homogeneous boundary equations. Thus we can always solve the system, while specifying one amplitude (say that of the incident wave in the half-space) arbitrarily. The solution describes the multiple reflection of an incident real SH-wave by the set of layers.

Now, if $\kappa > \omega/\beta$, we have a single permitted complex SH-wave in the half-space and two waves in each layer. The boundary conditions reduce to $2n + 1$ homogeneous equations in $2n + 1$ unknowns, and solutions are possible only if the determinant of

this system of equations vanishes. This condition is called the "period equation"; its solution specifies ω as a multivalued function of κ. Thus we have a series of waveguide modes, called Love waves, each characterized by a dispersion relation (solution of the "period equation") giving ω as a function of κ. From this relation we can derive the equations for phase velocity (ω/κ) and group velocity ($d\omega/d\kappa$) as functions of κ.

For the case of a single layer, the behavior of Love waves is fairly simple. They can exist only if the shear velocity (β) in the half-space exceeds that (β') in the layer. Also, for a given frequency, ω, the system determinant has zeros only for values of κ in the range

$$\frac{\omega}{\beta} \leq \kappa \leq \frac{\omega}{\beta'} \tag{4.1}$$

Thus, on the SH ω-κ diagram, the dispersion curves, which also represent the loci of surface-wave eigenfunctions, lie entirely between the lines labelled β and β' in Fig. 7. In this wedge-shaped region the waves in the layer are real, while those in the half-space are complex. The period equation admits an infinite sequence of modes (each corresponding to an integral number of nodal planes in the layer), each confined to a dispersion curve of ω versus κ, as shown in Fig. 7. The first mode exists for all frequencies, and the higher modes exist for frequencies above the cut-off frequencies:

$$f_n = \frac{\omega_n}{2\pi} = \frac{n-1}{2} (\beta/H) \left[(\beta/\beta')^2 - 1 \right]^{-1/2} \tag{4.2}$$

where H is the thickness of the layer. Each mode propagates with phase velocity β near cut-off, and with phase velocity β' as ω becomes infinite. In other words, the phase velocity is that of the layer for wavelengths small compared to H, and that of the half-space in the opposite case.

The behavior of P—SV-waves is more complicated in detail. We now have four waves in each layer and four continuity equations at each interface. Again there are solutions in the continuum portion of the ω-κ diagram, representing incident P- or SV-waves and reflected waves of both types. There is also the region where P-waves are complex and SV-waves real in the half-space, as before. For values of κ greater than ω/β, both waves in the half-space are complex, and the boundary conditions for n layers reduce to 4n + 2 homogeneous equations in 4n + 2 unknown wave amplitudes. The resulting period equation again yields an infinite series of modes, each with its characteristic

equation of ω versus $\varkappa$. For a single layer, we find modes entering in pairs as ω increases past successive cut-off values, and all the dispersion relations are curves in the ω-$\varkappa$ plane. Hence Rayleigh-wave propagation is now dispersive, like Love-wave propagation, so that the non-dispersive Rayleigh waves of the pure half-space appear as a very special case. Typical curves are sketched qualitatively in Fig. 8 for P—SV-waves in a medium consisting of a single layer over a half-space. The three straight lines are labelled by their slopes, α, β, and c_R, equal to the phase velocities of P, S and Rayleigh waves, respectively, in the half-space. (The indicated asymptotic slopes are β', the shear velocity in the layer, and c_R', the Rayleigh wave velocity in the layer.)

Up to this point, we have considered only eigenfunctions made up of plane wave functions proportional to

$$e^{i(\vec{\varkappa}\cdot\vec{\rho}-\omega t)}$$

for real values of $\vec{\varkappa}$ and ω. The z-dependence is then either trigonometric or exponential, depending on the relative magnitudes of $\varkappa$ and ω. No other solutions are possible which are finite throughout all space (or half-space) and for all time. However, the wave equation (for P-waves, for example) is satisfied by the plane wave

$$\vec{u}_P(\vec{r},t) = \vec{U}_P\, e^{i(k_3 z+\vec{\varkappa}\cdot\vec{\rho}-\omega t)} \tag{4.3}$$

for complex values of k_3, $\vec{\varkappa}$, and ω, so long as

$$k_3^2 + \varkappa^2 - (\omega/\alpha)^2 = 0 \tag{4.4}$$

For the layered half-space problem it is found that the boundary conditions can still be satisfied for combinations of waves with real ω and complex $\vec{\varkappa}$. To put it another way, the same conditions (vanishing determinant) which yield the dispersion relations for Love and Rayleigh waves admit solutions for real ω and complex $\vec{\varkappa}$.

In order to see the physical nature of such waves, suppose that $\vec{\varkappa}$ is in the x-direction and that

$$\varkappa = a + ib$$

where a and b are real. Then (unless a = 0) $\varkappa^2$ is complex, and (4.4) implies that k_3^2, and hence k_3 itself, is complex. If we put

$$k_3 = c + id$$

then the plane wave (4.3) becomes

$$\vec{u}_P(\vec{r}, t) = \vec{U}_P \, e^{-dz + icz - bx + iax - i\omega t} \tag{4.5}$$

Formula (4.5) describes a complex plane wave of a more general type than those considered before. The disturbance propagates, with attenuation, in both x and z directions. It turns out that solutions exist for positive values of a, b, and c, and negative d. Thus energy travels along the surface (positive x direction) with exponentially decreasing amplitude, while energy is also propagating into the half-space (positive z direction) with exponentially increasing amplitude. These waves are not true surface waves because of the positive flux of energy out of the surface-layer waveguide into the half-space, and for this reason are called leaky modes. Such waves obviously cannot exist in the steady state (real ω) because they have unbounded amplitude. However, they have a definite place in transient wave propagation phenomena in which they enter the general eigenfunction expansion with complex values of ω. These modes may be indicated on the ω-$\varkappa$ diagram (for real ω) by plotting ω versus the real part of $\varkappa$. They appear as a series of dispersion curves located in the continuum portion of the diagram.

V. SIGNAL AND NOISE MODELS

Let us suppose, at first, that we require signal and noise models suitable for the study of a surface array of vertical-component sensors. The vertical component of earth displacement, u_z, as a function of position on the surface, $\vec{\rho}$, and time, t, can be written in the form

$$u_z(\vec{\rho},t) = \iiint_{-\infty}^{\infty} \psi(\omega,\vec{\kappa})\, e^{i(\vec{\kappa}\cdot\vec{\rho}-\omega t)}\, d\vec{\kappa}\, d\omega \tag{5.1}$$

Formula (5.1) is introduced here simply as a three-dimensional Fourier transform, where frequency (ω) is conjugate to time, and the components of horizontal wave number ($\vec{\kappa}$) are conjugate to the horizontal position coordinates. Since u_z is a real quantity, it follows that

$$\psi(-\omega,-\vec{\kappa}) = \overline{\psi(\omega,\vec{\kappa})} \tag{5.2}$$

where the bar denotes complex conjugate. As a result of (5.2) we can rewrite (5.1) in the form

$$u_z(\vec{\rho},t) = \iint_{-\infty}^{\infty} d\vec{\kappa} \int_0^{\infty} d\omega\, \{X'(\omega,\vec{\kappa}) \cos(\vec{\kappa}\cdot\vec{\rho}-\omega t) - Y'(\omega,\vec{\kappa}) \sin(\vec{\kappa}\cdot\vec{\rho}-\omega t)\} \tag{5.3}$$

where X' and Y' are real, and

$$X'(\omega,\vec{\kappa}) + i\, Y'(\omega,\vec{\kappa}) = 2\psi(\omega,\vec{\kappa}) \tag{5.4}$$

If we use polar coordinates in the $\vec{\kappa}$-plane, so that κ is the length of $\vec{\kappa}$ and α is the polar angle, and also polar coordinates in space, with $\rho = |\vec{\rho}|$ and azimuth angle θ, then (5.3) can be written

$$u_z(\rho,\theta,t) = \int_0^{2\pi} d\theta \iint_0^{\infty} d\omega\, d\kappa\, \{X(\omega,\kappa,\alpha) \cos[\kappa\rho\cos(\alpha-\theta) - \omega t]$$

$$- Y(\omega,\kappa,\alpha) \sin[\kappa\rho\cos(\alpha-\theta) - \omega t]\} \tag{5.5}$$

where $X(\omega,\kappa,\alpha) = \kappa X'(\omega,\vec{\kappa})$ and $Y(\omega,\kappa,\alpha) = \kappa Y'(\omega,\vec{\kappa})$.

Now that the field is represented as an integral over azimuth (α) in the first quadrant of the ω-κ plane, we may interpret X and Y as amplitudes in the same ω-κ space discussed in earlier sections. In other words, form (5.5) makes it possible to inject into the model such physical assumptions as "signal is P-wave only" or "noise is all in the first Rayleigh mode" etc., simply by limiting non-zero values of X and Y to appropriate portions of the ω-κ space. Generally, signals contain many phases, and noise is present in several surface-wave modes, hence adequate models will require amplitudes (X and Y) which are sums of several terms, each relatively sharply confined to a portion of the ω-κ space.

The representation (5.5) is directly suitable to model deterministic signals of known or unknown form. However, to describe noise, or to derive a random model for signals, we begin with a statistical specification and proceed to derive an analog of (5.5). Let us assume that u_z is a random function of $\vec{\rho}$ and t with mean zero:

$$E\, u_z(\vec{\rho},t) = 0 \tag{5.6}$$

The symbol E stands for expectation, or ensemble average. We also assume that the noise is (wide-sense) stationary in both space and time, so that its covariance function, Γ, has the form

$$E\, u_z(\vec{\rho},t)\, u_z(\vec{\rho}\,',t') = \Gamma(\vec{\rho} - \vec{\rho}\,',\ t - t') \tag{5.7}$$

(The assumption of stationarity is a convenience, not a necessity; very similar results can be obtained without its use, but the analysis is more complex.) The covariance function, $\Gamma(\vec{\rho},t)$, always has a spectrum; we assume it has a spectral density, $G'(\omega,\vec{\kappa})$

$$\Gamma(\vec{\rho},t) = \iiint_{-\infty}^{\infty} G'(\omega,\vec{\kappa})\ e^{i(\vec{\kappa}\cdot\vec{\rho}-\omega t)}\ d\omega\, d\vec{\kappa} \tag{5.8}$$

As before, this relation is a purely mathematical one, essentially a Fourier transform, and we choose to call the variables conjugate to $\vec{\rho}$ and t by the names $\vec{\kappa}$ and ω, respectively.

Because $\Gamma(\vec{\rho},t)$ is a covariance, $G'(\omega,\vec{\varkappa})$ is real and non-negative. Because u_z, and hence $\Gamma(\vec{\rho},t)$ is real, we have

$$G'(-\omega,-\vec{\varkappa}) = G'(\omega,\vec{\varkappa}) \tag{5.9}$$

This symmetry allows us to write (5.8) as an integral over positive frequencies only:

$$\Gamma(\vec{\rho},t) = 2\int_0^\infty d\omega \iint_{-\infty}^{\infty} d\vec{\varkappa}\; G'(\omega,\vec{\varkappa}) \cos(\vec{\varkappa}\cdot\vec{\rho}-\omega t) \tag{5.10}$$

Finally, in polar coordinates, we put

$$G(\omega,\varkappa,\alpha) = 2\varkappa\, G'(\omega,\vec{\varkappa})$$

and find

$$\Gamma(\rho,\theta,t) = \int_0^{2\pi} d\alpha \iint_0^{\infty} d\omega\, d\varkappa\;\; G(\omega,\varkappa,\alpha)\cos[\varkappa\rho\cos(\alpha-\theta)-\omega t] \tag{5.11}$$

The well-known spectral representation theorem for stationary random functions of time has a direct analog for stationary random functions of several variables. When applied to the random function $u_z(\vec{\rho},t)$ and expressed in polar coordinates, the representation is

$$u_z(\rho,\theta,t) = \int_0^{2\pi} \iint_0^{\infty} \{\cos[\varkappa\rho\cos(\alpha-\theta)-\omega t]\; d\xi(\omega,\varkappa,\alpha)$$

$$-\sin[\varkappa\rho\cos(\alpha-\theta)-\omega t]\; d\eta(\omega,\varkappa,\alpha)\} \tag{5.12}$$

In (5.12), the processes $\xi(\omega,\varkappa,\alpha)$ and $\eta(\omega,\varkappa,\alpha)$ are uncorrelated, zero-mean processes with orthogonal increments. These increments have variances given by

$$E[d\xi(\omega,\varkappa,\alpha)]^2 = E[d\eta(\omega,\varkappa,\alpha)]^2 = G(\omega,\varkappa,\alpha)\, d\omega\, d\varkappa\, d\alpha \tag{5.13}$$

Heuristically, we can continue to use the Fourier transform (5. 5) to represent noise, if we interpret $X(\omega, \varkappa, \alpha)$ and $Y(\omega, \varkappa, \alpha)$ as uncorrelated random functions of zero mean satisfying

$$E\, X(\omega, \varkappa, \alpha)\, X(\omega', \varkappa', \alpha') = E\, Y(\omega, \varkappa, \alpha)\, Y(\omega', \varkappa', \alpha')$$

$$= G(\omega, \varkappa, \alpha)\, \delta(\omega' - \omega)\, \delta(\varkappa' - \varkappa)\, \delta(\alpha' - \alpha) \tag{5. 14}$$

If we wish to model the noise as a superposition of surface-wave modes (uncorrelated), we may replace $G(\omega, \varkappa, \alpha)$ by a sum of terms, each vanishing off the corresponding dispersion curve of the mode in ω-$\varkappa$ space. Since these separate terms in $G(\omega, \varkappa, \alpha)$ will not overlap, the representation (5. 12) (or (5. 5)) may be written as a sum of terms, satisfying equations like (5. 13) or (5. 14) for each mode.

According to (5. 11), any assumed power spectral density in ω-$\varkappa$ space determines the correlation properties of the noise in space and time. For example, if the noise power is isotropic, then $G(\omega, \varkappa, \alpha) \equiv G_0(\omega, \varkappa)/2\pi$, is independent of α and we can perform the α-integration. The result is

$$\Gamma(\rho, \theta, t) = \iint_0^\infty G_0(\omega, \varkappa)\, J_0(\varkappa\rho) \cos \omega t\, d\omega\, d\varkappa \tag{5. 15}$$

where J_0 is the Bessel function of zero order. Note that the correlation function is also isotropic. If, further, this isotropic noise is propagating in a single surface-wave mode, we can express the dispersion curve for the mode in the form $\varkappa = \varkappa(\omega)$. Then G_0 will vanish off this curve:

$$G_0(\omega, \varkappa) = P(\omega)\, \delta[\varkappa - \varkappa(\omega)]$$

and we obtain the single integral

$$\Gamma(\rho, t) = \int_0^\infty P(\omega)\, J_0[\rho\varkappa(\omega)] \cos \omega t\, d\omega \tag{5. 16}$$

Since $J_0(0) = 1$, we see that $P(\omega)$ is the one-sided spectral density (per radian) of the noise at a given location, and that the cross-spectral density for the noise waveforms

at two points a distance ρ apart is completely fixed by $P(\omega)$ and the assumptions of isotropy and single-mode dispersion.

More generally, we may expand the spectral density as a Fourier series in azimuth:

$$G(\omega, \kappa, \alpha) = \frac{1}{2\pi} G_o(\omega, \kappa) + \frac{1}{2\pi} \sum_{n=1}^{\infty} [G_n^+(\omega, \kappa) \cos n\alpha + G_n^-(\omega, \kappa) \sin n\alpha] \tag{5.17}$$

When substituted in (5.11), we can perform the α-integration to obtain the correlation function as a Fourier series in θ:

$$\begin{aligned}\Gamma(\rho, \theta, t) &= \iint_0^\infty G_o(\omega, \kappa) J_o(\kappa\rho) \cos \omega t \, d\omega \, d\kappa \\ &+ \sum_{n=1} \cos n\theta \iint_0^\infty G_n^+(\omega, \kappa) J_n(\kappa\rho) \cos (\omega t - \frac{n\pi}{2}) \, d\omega \, d\kappa \\ &+ \sum_{n=1} \sin n\theta \iint_0^\infty G_n^-(\omega, \kappa) J_n(\kappa\rho) \cos (\omega t - \frac{n\pi}{2}) \, d\omega \, d\kappa\end{aligned} \tag{5.18}$$

Again, the double integrals may be reduced to single integrals (or a sum of single integrals) by the assumption of a single mode (or a sum of modes).

The processing of the output signals of an array can be discussed in terms of processing the elastic field itself. For example, a general linear functional of the field has the form

$$\iiint_{-\infty}^{\infty} f(\vec{\rho}, t) \, u_z(\vec{\rho}, t) \, d\vec{\rho} \, dt \tag{5.19}$$

for some fixed function f. An actual array does not sample the field continuously in space, but discretely, and thus is described by an f-function which is a sum of delta-functions in the space variable. However, it is easier to see what is happening if we use space integrals instead of sums over sensors. Formula (5.19) describes a single number based on the field for all values of time. Actual processors provide a function

of real time, based on the field for the past only (i.e., they use realizable filters). If the output of such a linear processor is called y(t), we may write

$$y(t) = \iint_{-\infty}^{\infty} d\vec{\rho} \int_{0}^{\infty} ds\, h(\vec{\rho}, s)\, u_z(\vec{\rho}, t-s) \tag{5.20}$$

where the function, $h(\rho, s)$, completely describes the processor. In practice, many such linear processors may operate, in parallel, on the same input field, and their outputs may be combined in a non-linear manner. However, we shall analyze only the linear aspect of the processing. For an array of sensors at points $\vec{\rho} = \vec{\rho}_n$, $n = 1, 2, \ldots, N$, we have

$$h(\vec{\rho}, t) = \sum_{n=1}^{N} h_n(t)\, \delta(\vec{\rho} - \vec{\rho}_n) \tag{5.21}$$

where $h_n(t)$ is the impulse response of the filter applied to the signal at the n'th sensor (including the response of the seismometer itself).

In order to express the processing operation (5.20) in ω-κ space, we use the original representation (5.1), which is related to (5.5) by means of

$$\psi(\omega, \vec{\kappa}) = \frac{1}{2\kappa} \{X(\omega, \kappa, \alpha) + i\, Y(\omega, \kappa, \alpha)\} \tag{5.22}$$

Property (5.2) is accounted for by defining X to be even in ω, and Y to be odd in ω. Formula (5.1) can describe noise, as well as signal, if we employ Eq. (5.14). Now if we substitute (5.1) in (5.20), we find

$$y(t) = \iiint_{-\infty}^{\infty} H(\omega, \vec{\kappa})\, \psi(\omega, \vec{\kappa})\, e^{-i\omega t}\, d\omega\, d\vec{\kappa} \tag{5.23}$$

where

$$H(\omega, \vec{\kappa}) = \iint_{-\infty}^{\infty} \int_{0}^{\infty} h(\vec{\rho}, s)\, e^{i(\vec{\kappa} \cdot \vec{\rho} + \omega s)}\, d\vec{\rho}\, ds \tag{5.24}$$

Since $h(\vec{\rho}, s)$ is real, $H(-\omega, -\vec{\kappa}) = \overline{H(\omega, \vec{\kappa})}$, and (5.23) may be written

$$y(t) = 2\,\mathrm{Re} \int_0^{2\pi} d\alpha \iint_0^{\infty} \varkappa\, d\omega\, d\varkappa\, H(\omega,\varkappa,\alpha)\, \psi(\omega,\varkappa,\alpha)\, e^{-i\omega t} \tag{5.25}$$

Thus $H(\omega,\varkappa,\alpha)$, for values of ω and $\varkappa$ in the first quadrant of the ω-$\varkappa$ plane, completely determines the processing. For the array mentioned above, we have

$$H(\omega,\vec{\varkappa}) = \sum_{n=1}^{N} e^{i\vec{\varkappa}\cdot\rho_n}\, H_n(\omega) \tag{5.26}$$

where

$$H_n(\omega) = \int_0^{\infty} h_n(s)\, e^{i\omega s}\, ds \tag{5.27}$$

is the frequency response of the filter, $h_n(t)$.

Formula (5.25) may be taken as the starting point in the design of a linear processor. The desired response in ω-$\varkappa$ space determines $H(\omega,\varkappa,\alpha)$, which then determines $h(\vec{\rho},t)$ by means of the inverse of (5.24)

$$h(\vec{\rho},t) = (2\pi)^{-3} \iiint_{-\infty}^{\infty} H(\omega,\vec{\varkappa})\, e^{-i(\vec{\varkappa}\cdot\vec{\rho}+\omega t)}\, d\omega\, d\vec{\varkappa}$$

$$= 2\,\mathrm{Re}\,(2\pi)^{-3} \int_0^{2\pi} d\alpha \iint_0^{\infty} d\omega\, d\varkappa\, \varkappa H(\omega,\varkappa,\alpha)\, e^{-i\varkappa\rho\cos(\alpha-\theta) - i\omega t} \tag{5.28}$$

The function $h(\vec{\rho},t)$ is then approximated by a discrete sum of the form (5.21), and the resulting processor evaluated for performance, aliasing introduced by discrete space-sampling, etc. Many equivalent procedures also suggest themselves.

As an illustration, suppose we wish to sense only surface waves of a single mode, described by a dispersion curve $\varkappa = \varkappa(\omega)$, in a given frequency band. Then we may put (for $\omega \geq 0$ and $\varkappa \geq 0$)

$$H(\omega,\varkappa,\alpha) = \frac{1}{\varkappa}\, F(\omega)\, \delta[\varkappa - \varkappa(\omega)] \tag{5.29}$$

This will accept such waves isotropically, with a frequency weighting function $F(\omega)$. When we put (5.29) into (5.28), we find

$$h(\vec{\rho},t) = (2\pi)^{-2} \int_{-\infty}^{\infty} F(\omega)\ J_0[\rho\varkappa(\omega)]\ e^{-i\omega t}\,d\omega\ , \tag{5.30}$$

where $F(-\omega) \equiv \overline{F(\omega)}$. The processing is isotropic, with a different filter for each radial distance. For example, an array might be arranged in concentric rings of sensors, all elements of a ring being added and passed through a filter with frequency response

$$F(\omega)\ J_0[\rho\varkappa(\omega)]\ , \tag{5.31}$$

where ρ is the radius of the ring. As a special case, to pick out a particular phase velocity, c, we would use (5.31) with $\varkappa(\omega) = \omega/c$. *

If we wish to think of the processing in terms of frequency and phase velocity, we simply eliminate $\varkappa$ by a change of variables:

$$\varkappa = \omega/c \tag{5.32}$$

where $0 \leq c < \infty$, and introduce

$$H_1(\omega,c,\alpha) \equiv \frac{\omega^2}{c^3}\ H(\omega, \frac{\omega}{c}, \alpha) \tag{5.33}$$

Then $h(\vec{\rho},t)$ is given by

$$h(\vec{\rho},t) = 2\ \mathrm{Re}\ (2\pi)^{-3} \int_0^{2\pi} d\alpha \iint_0^{\infty} d\omega\, dc\ H_1(\omega,c,\alpha)\ e^{-i(\omega/c)\cos(\alpha-\theta) - i\omega t} \tag{5.34}$$

For example, if we make H_1 independent of α and proportional to $\delta(c - c_0)$, we return to (5.31) with $\varkappa(\omega) = \omega/c_0$.

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* Processing schemes which accept waves within a fixed range of phase velocities have been designed and implemented (Ref. 3). They have proved quite useful in reducing microseismic noise.

These models of one component of earth motion can easily be extended to three-component models of surface motion. The displacement is now a vector, $\vec{u}(\vec{\rho},t)$, hence (5.1) is generalized to

$$\vec{u}(\vec{\rho},t) = \iiint_{-\infty}^{\infty} \vec{\psi}(\omega,\vec{\varkappa})\ e^{i(\vec{\varkappa}\cdot\vec{\rho}-\omega t)}\ d\omega\, d\vec{\varkappa} \tag{5.35}$$

where the transform, $\vec{\psi}$, is also a vector. We think of a plane-layered model earth, and draw the ω-$\varkappa$ diagram for the underlying half-space, keeping in mind that (5.35) describes the motion at the upper (free) surface of the first layer. Using the results of the previous sections, we know that eigenfunctions are possible for all points of the continuum portion of the ω-$\varkappa$ space, hence $\vec{\psi}(\omega,\vec{\varkappa})$ need not vanish anywhere in this portion. For fixed ω and $\vec{\varkappa}$, the component of $\vec{\psi}$ in the surface and perpendicular to $\vec{\varkappa}$ obviously represents the amplitude of an SH-wave, and we know that there is just one such eigenfunction (of arbitrary amplitude) for each point $(\omega,\varkappa)$. The other two components of $\vec{\psi}(\omega,\vec{\varkappa})$ describe P—SV-waves, and we know that there are two independent P-SV eigenfunctions for each point $(\omega,\varkappa)$. Thus we may choose these two components of $\vec{\psi}$ arbitrarily, and we have shown that any vector amplitude function, $\vec{\psi}(\omega,\vec{\varkappa})$ is possible for the continuum portion of the ω-$\varkappa$ space. In the surface-wave portion of the ω-$\varkappa$ plane, $\vec{\psi}$ must vanish except along the curves describing surface-wave modes. For each such mode, the polarization is a given function of ω and α, hence only a scalar function of frequency can be assigned arbitrarily as the amplitude of each mode.

In a similar fashion, the model can be extended in depth to cover the general case, by writing

$$\vec{u}(\vec{\rho},z,t) = \iiint_{-\infty}^{\infty} \vec{\psi}(\omega,\vec{\varkappa},z)\ e^{i(\vec{\varkappa}\cdot\vec{\rho}-\omega t)}\ d\omega\, d\vec{\varkappa} \tag{5.36}$$

For each eigenfunction, the z-dependence is fixed, hence it is not appropriate to add a fourth variable of integration conjugate to z, and no new degrees of freedom are present over the case $z = 0$. Thus a statistical specification of the mode structure of the noise automatically determines the correlation properties in depth, and polarization, as well as in surface location and time, according to an assumed model of the earth.

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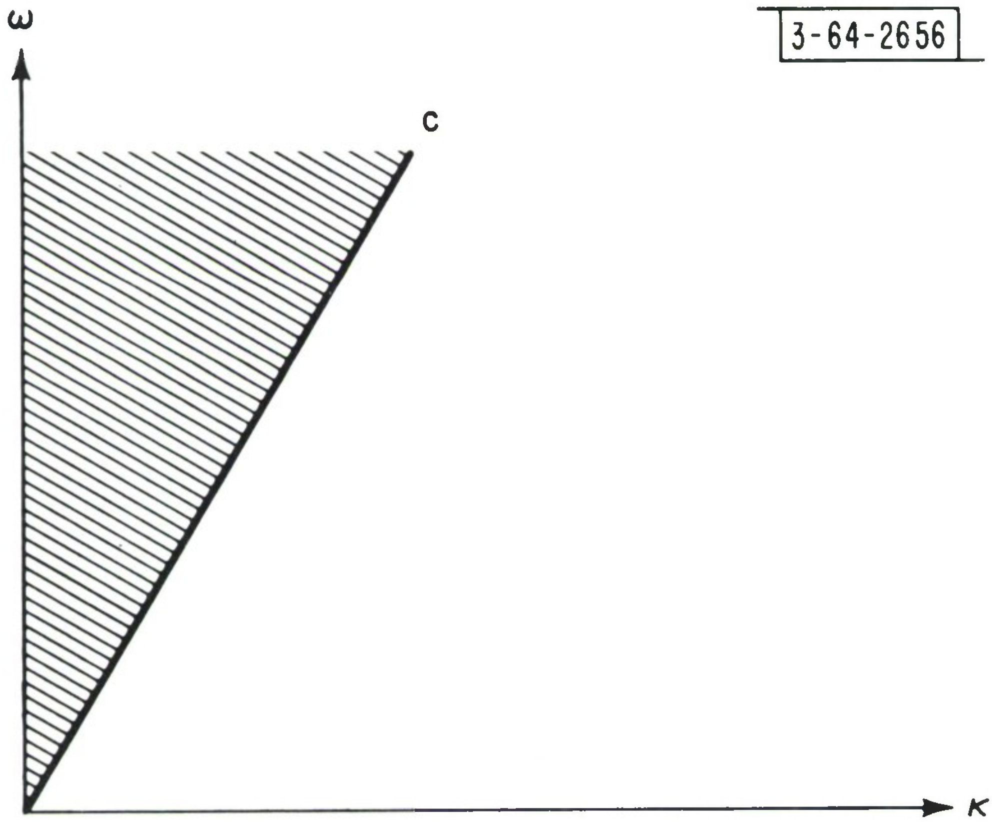

Fig. 1

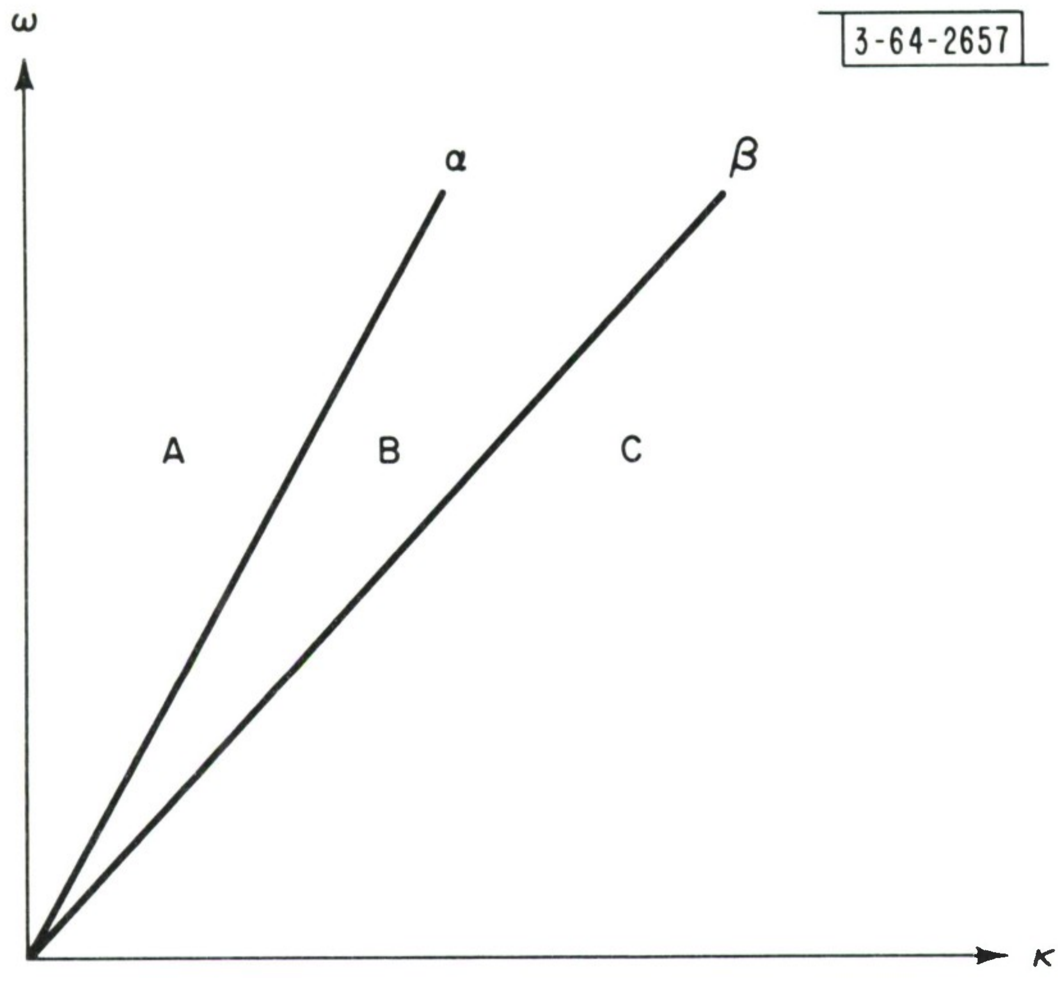

Fig. 2

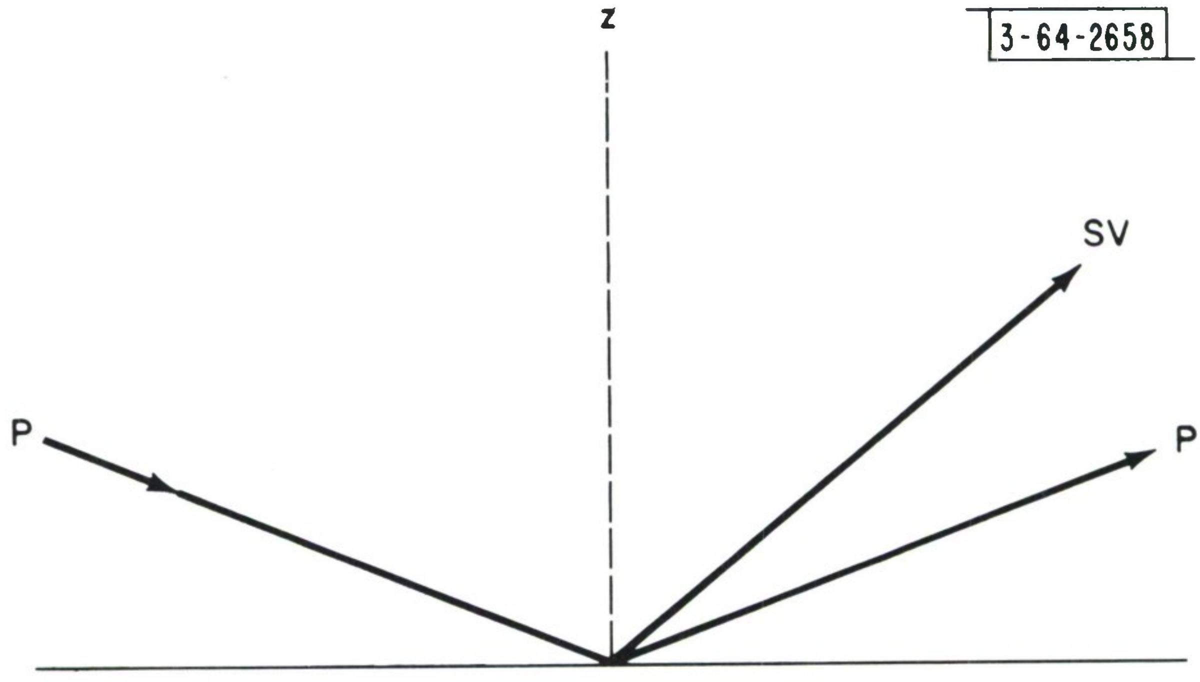

Fig. 3

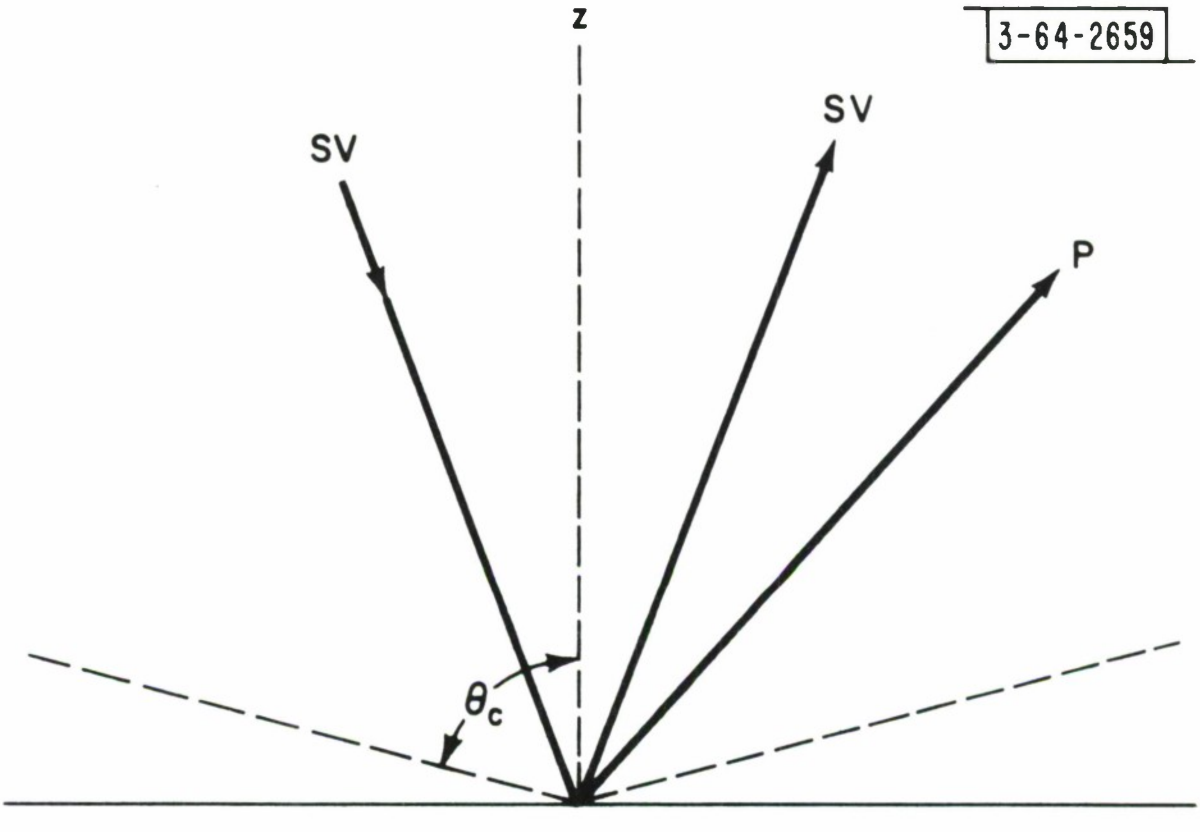

Fig. 4

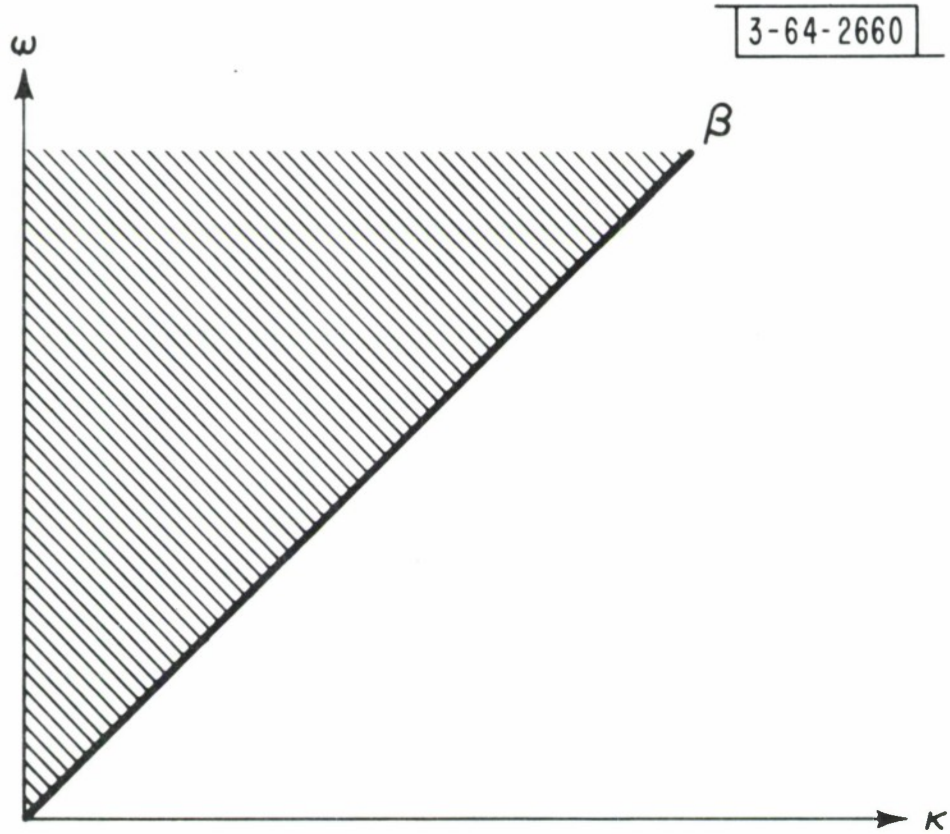

Fig. 5

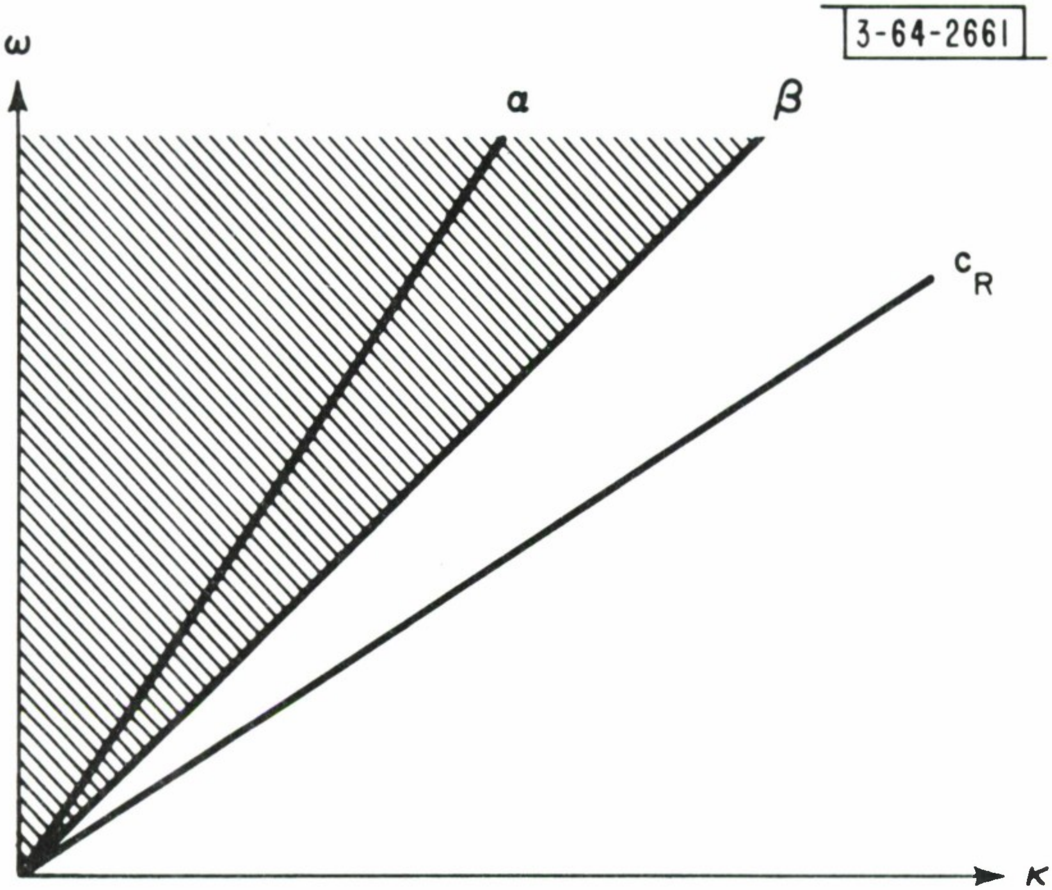

Fig. 6

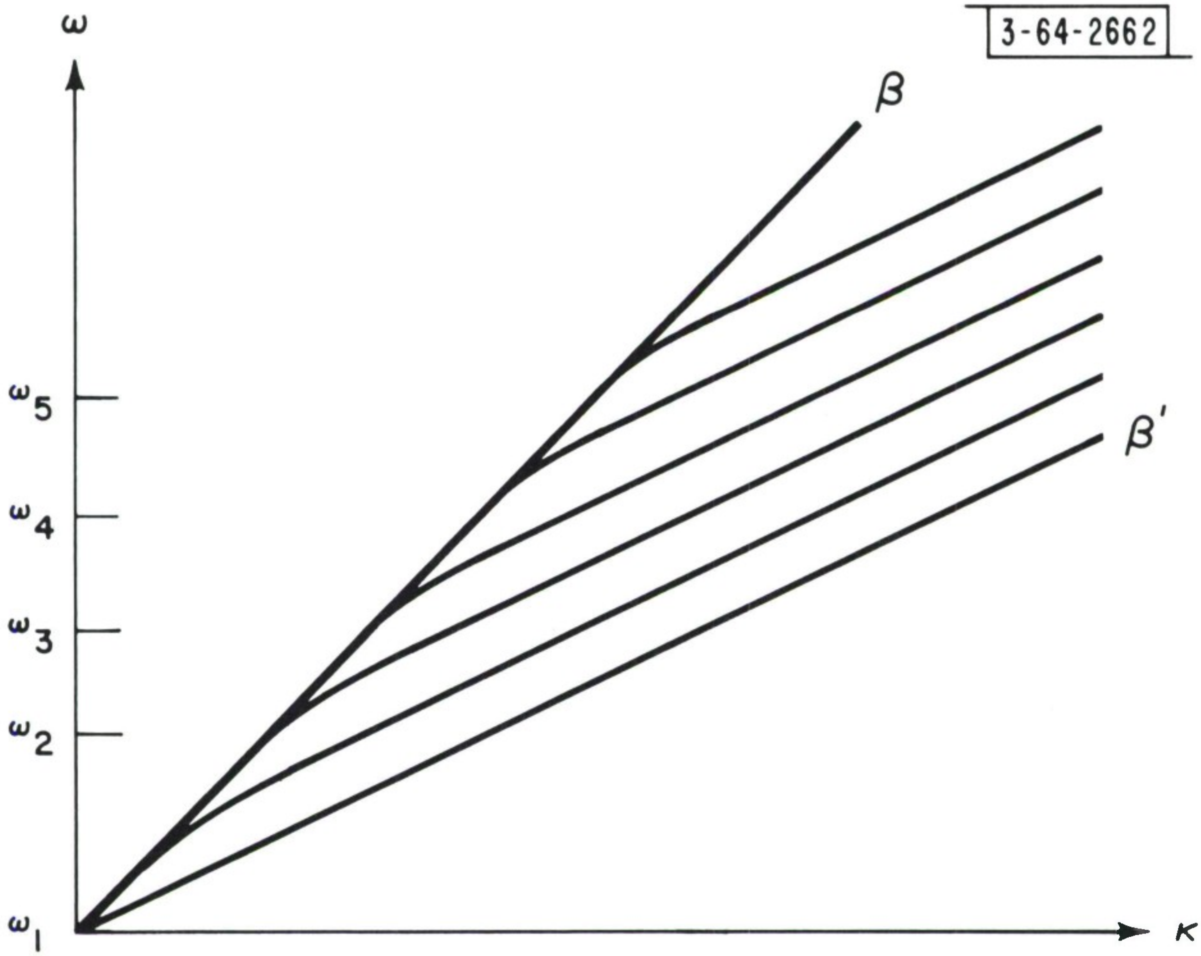

Fig. 7

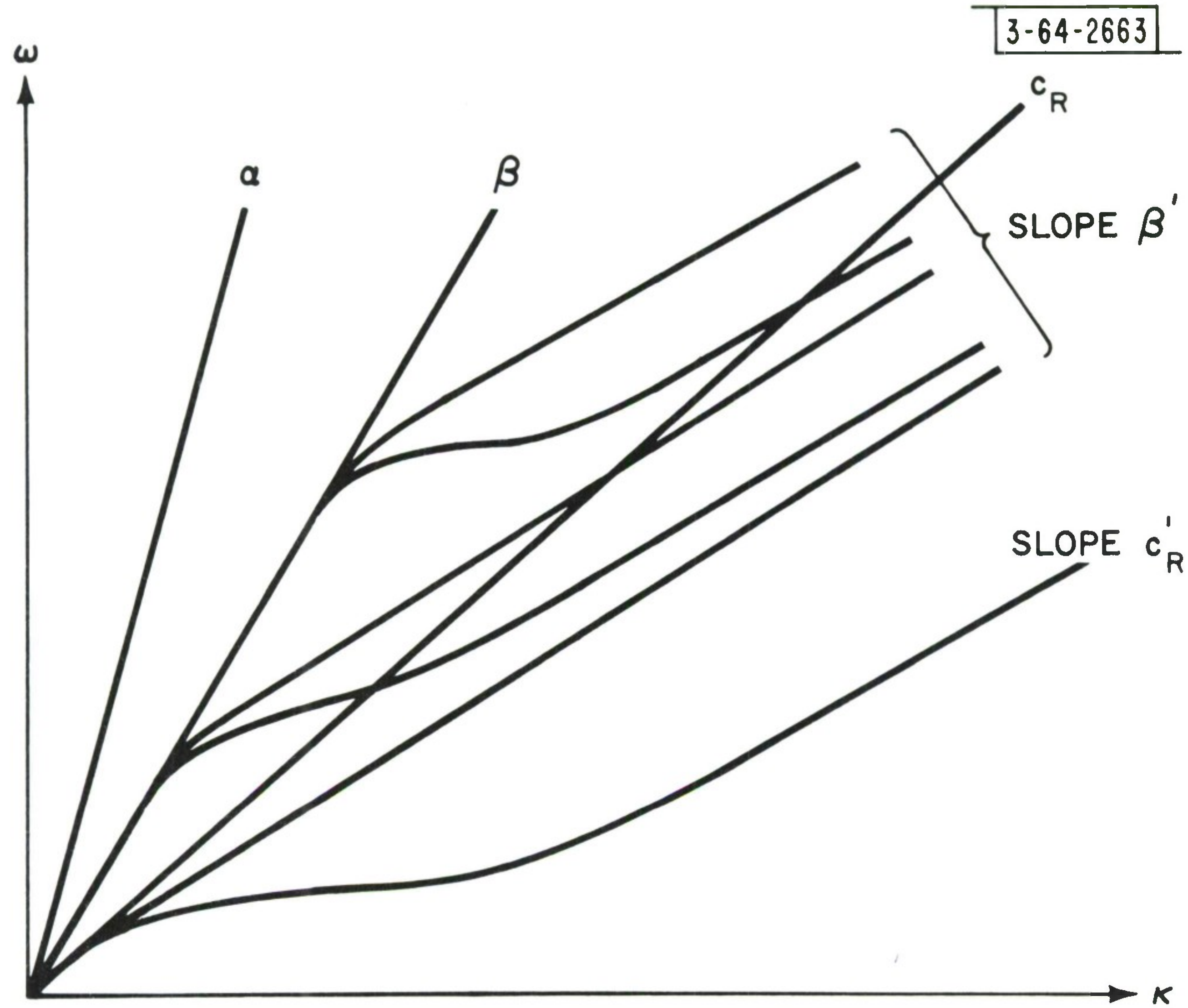

Fig. 8